New evidence for the infinitude of some prime constellations

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ABSTRACT

A conjecture of Hardy and Littlewood, implying the twin-primes conjecture and featured in the work of Arenstorf, is extended to some other prime constellations. Computational evidence is presented in support of the conjectures, which imply the infinitude of these constellations, including the twin primes.

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Prime constellations, twin-primes conjecture, prime triplets, prime quadruplets, prime k-tuples conjecture, Hardy-Littlewood approximations.

EXPOSITION

One of the famous unsettled questions in number theory is the "twin-primes conjecture": there exists an infinite number of primes p for which $p+2$ is also prime. R. F. Arenstorf [1] recently set forth a proposed proof of the twin-primes conjecture, employing analytic continuation, the complex Tauberian theorem of Wiener and Ikehara, and other advanced techniques of classical analytic number theory. Following the discovery of a significant error [8], the paper was withdrawn for revision. However, the genesis of the present paper is in Arenstorf's strategy, which was based on an attempt to prove a related stronger conjecture of Hardy and Littlewood [4],

$$(1) \qquad \lim_{n\to\infty} F_2(n) = 2c_2 = 1.32032363169373914785562422002911155686524672\ldots \quad ,$$

where

$$(2) \qquad F_2(n) = \frac{1}{n}\sum_{p\leq n} \ln p \cdot \ln(p+2) \quad ,$$

the sum being taken over all twin-prime pairs $(p, p+2)$ with $p \leq n$ (Arenstorf uses $p < n$, but the difference is trivial, and the convention here is the one adopted in several previous studies of the twin primes). The constant c_2 is the twin-primes constant, defined as

$$(3) \qquad c_2 = \prod_{p>2} \frac{p(p-2)}{(p-1)^2} = \prod_{p>2}\left[1 - \frac{1}{(p-1)^2}\right] = 0.660161815846869573927812110\ldots \quad ,$$

the infinite products being taken over all primes except 2; c_2 was computed to 42 decimals by Wrench [10], and eventually to more than 1000 decimals by Moree and Niklasch [5]. Note that some authors refer to $2c_2$ as the twin-primes constant.

Proving that the limit in (1) is $2c_2$, or indeed any positive constant, is tantamount to proving the twin-primes conjecture; for if the twin primes were finite in number, the sum in (2) would necessarily be bounded, and the limit would be zero. This is in contrast to the sum of the reciprocals of the twin primes, which was proven [2] to converge to "Brun's constant", $B_2 = 1.90216\ 05826\ 22 \pm 0.00000\ 00016\ 04$ [6], whether or not the twin primes are infinite in number.

In support of (1), Wolf [9] computed $F_2(n)$ for $n = 2^k$ up to $2^{40} \approx 1.1 \times 10^{12}$, and observed excellent convergence toward the predicted limit. Additional such numerical evidence is presented as part of this paper.

The twin-primes constellation consists of all prime pairs of the form $(p, p+2)$, beginning with $(3,5)$ and $(5,7)$. We will consider three additional prime constellations—two sets of prime triplets and a set of prime quadruplets. These are chosen for several reasons. The author has been studying them for some time in other contexts; they are in some sense among the simplest constellations other than the twins; and historically they were among the first constellations to be studied [3, 7]. We thus consider the set Q_{3a} of all prime triplets of the form $(p, p+2, p+6)$, beginning with $(5,7,11)$; the set Q_{3b} of all prime triplets of the form $(p, p+4, p+6)$, beginning with $(7,11,13)$; and the set Q_4 of all prime quadruplets of the form $(p, p+2, p+6, p+8)$, beginning with $(5,7,11,13)$. The corresponding counting functions, indicating the number of members of each set such that $p \leq n$, are denoted by $\pi_{3a}(n)$, $\pi_{3b}(n)$, and $\pi_4(n)$.

Each of these three prime constellations, like the twins, is *admissible*; that is, the members do not form a complete residue class with respect to any prime. In consequence, the prime k-tuples conjecture of Hardy and Littlewood [4] implies the infinitude of each of these constellations. A classic example of an *inadmissible* constellation is the set of prime triplets of the form $(p, p+2, p+4)$, which constitute a complete residue class modulo 3, and therefore must be finite in number (the only instance being $(3,5,7)$).

Hardy and Littlewood's prime k-tuples conjecture [4] furthermore implies specific (asymptotic) densities for admissible prime constellations, with consequent integral formulas (the Hardy-Littlewood approximations) approximating the counts of each such constellation; these are analogues of the Prime Number Theorem, $\pi(x) \sim \mathrm{Li}(x)$. Here we use the notation that $f(x)$ is *asymptotic* to $g(x)$ as $x \to \infty$, $f(x) \sim g(x)$, if and only if $\lim_{x\to\infty} f(x)/g(x) = 1$. In particular, the Hardy-Littlewood approximation for the count $\pi_2(n)$ of twin primes is [4, p. 44]

$$\pi_2(n) \sim 2c_2 \int_2^n \frac{dt}{\ln^2 t} \quad . \tag{4}$$

In fact, conjecture (1) can be deduced from conjecture (4), as follows. If dP represents the number of twin-prime pairs in an interval of measure dt,

$$\begin{aligned} F_2(n) &\sim \frac{1}{n} \int_{t=2}^{t=n} \ln t \cdot \ln(t+2)\ dP \\ &\sim \frac{1}{n} \int_{t=2}^{t=n} \ln^2 t\ dP \\ &\sim \frac{1}{n} \int_2^n \ln^2 t\, \frac{2c_2\ dt}{\ln^2 t} \quad , \qquad \text{by (4);} \\ &\sim \frac{1}{n} \int_2^n 2c_2\ dt \end{aligned} \tag{5}$$

$$
\begin{aligned}
F_2(n) &\sim \frac{1}{n}\int_0^n 2c_2\ dt \\
&\sim \frac{1}{n}2c_2 n \\
&\sim 2c_2 \quad ,
\end{aligned}
$$

from which (1) follows.

In a similar fashion, we can start from the Hardy-Littlewood approximations [4] for other admissible prime constellations and deduce conjectured relationships analogous to (1). Relevant to the three additional prime constellations we have chosen to consider, we thus define

$$
\text{(6a)} \qquad F_{3a}(n) = \frac{1}{n}\sum_{p\le n} \ln p \cdot \ln(p+2) \cdot \ln(p+6) \quad ,
$$

$$
\text{(6b)} \qquad F_{3b}(n) = \frac{1}{n}\sum_{p\le n} \ln p \cdot \ln(p+4) \cdot \ln(p+6) \quad ,
$$

$$
\text{(6c)} \qquad F_4(n) = \frac{1}{n}\sum_{p\le n} \ln p \cdot \ln(p+2) \cdot \ln(p+6) \cdot \ln(p+8) \quad .
$$

Here F_{3a} is evaluated over Q_{3a}, F_{3b} is evaluated over Q_{3b}, and F_4 is evaluated over Q_4. The Hardy-Littlewood integral approximations for these three constellations are as follows [7, p. 61].

$$
\text{(7a)} \qquad \pi_{3a}(n) \sim \tfrac{9}{2}c_3 \int_2^n \frac{dt}{\ln^3 t} \quad ,
$$

$$
\text{(7b)} \qquad \pi_{3b}(n) \sim \tfrac{9}{2}c_3 \int_2^n \frac{dt}{\ln^3 t} \quad ,
$$

$$
\text{(7c)} \qquad \pi_4(n) \sim \tfrac{27}{2}c_4 \int_2^n \frac{dt}{\ln^4 t} \quad .
$$

The constants c_3 and c_4 are additional instances (along with c_2) of the Hardy-Littlewood constants [4], defined in general for $k = 2, 3, 4, \ldots$, by an infinite product over primes,

$$
\text{(8)} \qquad c_k = \prod_{p>k} \frac{p^{k-1}(p-k)}{(p-1)^k} \quad .
$$

In particular,

$$
\text{(9a)} \qquad c_3 = \prod_{p>3} \frac{p^2(p-3)}{(p-1)^3} = 0.63516635460427120720669659127252241734 2\ldots \quad ,
$$

$$
\text{(9b)} \qquad c_4 = \prod_{p>4} \frac{p^3(p-4)}{(p-1)^4} = 0.307494878758327093123354486071076853\ldots \quad ,
$$

where the decimal approximations are taken from Moree and Niklasch [5], who have computed the Hardy-Littlewood and associated constants to 1002 decimals.

Derivations parallel to that of (5) now lead to the following new conjectures.

$$
\text{(10a)} \qquad \lim_{n\to\infty} F_{3a}(n) = \tfrac{9}{2}c_3 = 2.858248595719220432430134660726350878\ldots \quad ,
$$

(10b) $\lim_{n\to\infty} F_{3b}(n) = \frac{9}{2}c_3 = 2.858248595719220432430134660726350878\ldots$,

(10c) $\lim_{n\to\infty} F_4(n) = \frac{27}{2}c_4 = 4.1511808632374157571652855619595375157 9941\ldots$.

We remark that the aggregate constants appearing in (10) are denoted as D (in (10a) and (10b)) and E (in (10c) by Moree and Niklasch [5].

Note that, as with (1) and the twin-primes conjecture, conjectures (10) imply the infinitude of the associated prime constellations; indeed, the existence of any positive values for the limits would be sufficient, since the existence of only a finite number of a constellation would bound the corresponding sum in (6) and produce a limit of zero. Furthermore, any one of conjectures (10) would imply the twin-primes conjecture as well, since each triplet or quadruplet includes a twin-prime pair. In addition, conjectures (1) and (10) would also provide strong evidence in favor of the Hardy-Littlewood approximations and the prime k-tuples conjecture, of which they are consequences.

In support of conjectures (1) and (10), C code already designed for investigation of the prime constellations was modified to compute and record values of $F_2(n)$, $F_{3a}(n)$, $F_{3b}(n)$, and $F_4(n)$. The code was executed on personal computers using the Wintel environment and floating point arithmetic to long double precision (64-bit mantissa). Results to 10^{11} were checked by means of a second code, using another (slower) algorithm. Partial results are shown in Table 1, for $n = 10^k$, $k = 1, \ldots, 14$. The last line indicates the conjectured limiting values.

Table 1.

n	$F_2(n)$	$F_{3a}(n)$	$F_{3b}(n)$	$F_4(n)$
10^1	0.489996983652836	0.750978013334680	1.196828133529755	1.926220572770529
10^2	0.799745092944858	1.048692856826072	2.585524317018651	0.705708858983092
10^3	1.028368743410726	2.431181265229706	2.452935721305665	3.346148475381936
10^4	1.295058615919660	2.506112075929775	2.606434053615298	3.410982520765379
10^5	1.302244190054408	2.766307011496849	2.596941911928010	3.696922752013137
10^6	1.309190742738575	2.778357571785003	2.879280996080815	4.083968787517717
10^7	1.326032793752577	2.853365876050076	2.889851473019375	4.405342771818259
10^8	1.320064585311069	2.868713724586362	2.860412298570633	4.172717820117937
10^9	1.320027197802376	2.856981682913185	2.857859746634257	4.151690282083630
10^{10}	1.320400512636363	2.856403308005203	2.854899772709739	4.137502967905835
10^{11}	1.320369198427433	2.858844688347779	2.857182234184146	4.149169959064297
10^{12}	1.320341275524246	2.858611546644537	2.858473729549299	4.153135542402516
10^{13}	1.320329462705253	2.858318952277267	2.858405338766327	4.150776515382274
10^{14}	1.320324077892631	2.858246127485545	2.858260937723365	4.151257032735108
$+\infty$	1.320323631693739	2.858248595719220	2.858248595719220	4.151180863237416

The convergence of $F_2(n)$, $F_{3a}(n)$, $F_{3b}(n)$, and $F_4(n)$ toward the conjectured limits appears to be quite convincing; the values for $F_2(n)$ complement and are consistent with those of Wolf [9]. Of course, this does not constitute a proof of conjectures (1) and (10), but it does stand as persuasive evidence for them, and consequently for the infinitude of the corresponding prime constellations, as well as for the associated Hardy-Littlewood approximations and the prime k-tuples conjecture.

In regard to $F_2(n)$ and the definitive proof of the twin-primes conjecture, we await remediation of the deficiencies discovered in Arenstorf's work [1]. If this is accomplished, it may be possible to extend Arenstorf's technique to the other three constellations, and perhaps even to any admissible prime constellation, proceeding from straightforward modifications of (6) and (10).

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