Here is an example of an A paper from a similar class that we teach at the freshman level.

The write ups have exactly the same sections. The only difference is that your write-ups should be slightly more sophisticated and significantly more mathematical!!

Remember this paper was written by a freshman albeit a very good freshmen!!!

The Hall of Lights

Introduction: In this problem there is an extremely long hallway that contains 20,000 lights. Each light has a pull chain that will turn the light on and off. Outside the hallway is a line of 20,000 people that will walk through the hallway. Before the first person walks through all the lights are off. As the first person walks through they pull every chain, turning on every light. The 2nd person walks through and pulls every second chain, turning every even numbered light off. The 3rd person walks through pulling every third chain, turning some lights on and some lights off. This continues until the 20,000 person pull the 20,000 chain. Once everyone walks through which lights are on (or off)? I will call this the Hallway Light game in the remainder of this write up. When I first read this problem it sounded fun. My first guess was that the result would have something to do with the prime numbers.

Exploration: I did several types of examples to figure out what was going on. First I looked at examples with the primes. I checked the lights that were numbered 2, 3, 5, and 7, the first 4 primes. I noticed that light number 2's chain would be pulled by person 1 and person 2 then never again. Hence it would be off. Then light number 3's chain would be pulled by person 1 and person 3 then never again. Hence it would be off. In fact, I realized quickly that all the primes would be off because they would be turned on by the first person then off again by the person that had their number. I also noticed that the light numbered *n* would have its chain pulled once for every factor of *n*. For example light number 8 will be pulled by persons numbered 1 ,2, 4, and 8. Hence it would be **off**. It is clear that if a light's chain is pulled an even number of times it will be **off** and if it is pulled an odd number of times it will be **on**. Said another way, light number *n* will be **on** if it has an odd number of factors and **off** if it had an even number of factors. I thought that was the answer but Dr. Peterson said I had to figure out exactly which numbers had an odd number of factors.

To see if I could figure out a pattern I did the following, I played the Hallway Light game with 36 lights instead of 20,000. Here is the result:

1 on	2 off	3 off	4 on	5 off	6 off	7 off	8 off	9 on
10 off	11 off	12 off	13 off	14 off	15 off	16 on	17 off	18 of
19 off	20 off	21 off	22 off	23 off	24 off	25 on	26 off	27 off
28 off	29 off	30 off	31 off	32 off	33 off	34 off	35 off	36 on

From this example it the following is clear.

Conjecture: When playing the Hallway Light game with 20,000 lights only the lights that are numbered with a perfect square will be on. All other lights will be off. Said in mathematical terms, in the list positive integers only the perfect squares have an odd number of factors.

Proof: Let n a positive integer with the following list of factors (in order) $d_1, d_2, \ldots d_k$. Notice that the factors come in distinct pairs $n=d_1 \times d_k$, $n=d_2 \times d_{k-1}$, $n=d_3 \times d_{k-2}$, and so on, which means that there must be an even number of factors unless there is a factor, d_j, such that $n=d_j \times d_j$. Hence, n would be a perfect square.

Conclusion: I found it very interesting how this problem transitioned from the physical property of pulling a light's chain to the mathematical property of one number dividing another. In the future I would like to work on the following problems.

1. Imagine an extremely long hallway that contains 20,000 ceiling fans. Each fan has a pull chain that will cycle the fan through off, low speed, and high speed. Outside the hallway is a line of 20,000 people that will walk through the hallway. Before the first person walks through all the fans are off. As the first person walks through they pull every chain, turning every fan to low. The 2nd person walks through and pulls every second chain, turning every even numbered fan to high. The 3rd person walks through pulling every third chain, turning some fans to low, some fans to high, and other fans off. This continues until the 20,000 person pull the 20,000 chain. Once everyone walks through which fans are on low, high or off?

2. Imagine an extremely long hallway that contains 20,000 lights. Each light has a pull chain that will turn the light on and off. Outside the hallway is a line of many people that will walk through the hallway. Before the first person walks through all the lights are off. As the first person walks through they pull second chain, turning on every other light. The 2nd person walks through and pulls every fourth chain, turning every other even numbered light off. The 3rd person walks pulling every eighth chain, turning some lights on and some lights off. This continues until the n^{th} person pulls every $2^n < 20{,}000$ chain. Once everyone walks through which lights are on (or off)?

3. Imagine an extremely long hallway that contains 20,000 lights. Each pair of lights has a pull chain between them that will cycle the lights closest to the chain from on to off or from off to on. In this scenario each light is controlled by two different chains. Outside the hallway is a line of 10,000 people that will walk through the hallway. Before the first person walks through all the lights are off. As the first person walks through they pull every chain, turning on a bunch of lights. The 2nd person walks through and pulls every other chain. The 3rd person walks pulling every 3rd chain. This continues until the 10,000 th person pulls the 10,000 chain. Once everyone walks through which lights are on (or off)?