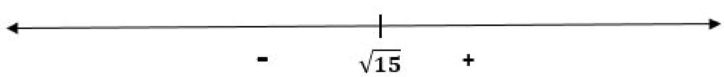
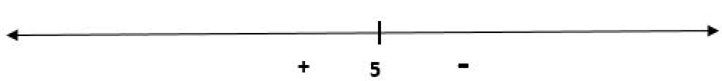
Optimization Problems!

1. Varying: The two numbers and the sum
   Constant: The product (15)
   Variable:
   Let x be one number
   Let y be the other
   Let S be the sum
   Formula for the quantity we are trying to minimize: $S = x + y$
   Formula for the constraint: $xy = 15$
   Solve the constraint for one of the variables: $y = \frac{15}{x}$.
   Write the equation for S using one variable using substitution: $S = x + \frac{15}{x}$
   We need to create the number line for S'. $S' = 1 - \frac{15}{x^2} = \frac{x^2 - 15}{x^2}$.
   Find where $S' = 0$ or where $x^2 - 15 = 0$. Hence $x = \pm\sqrt{15}$ ... but only $\sqrt{15}$ is non-negative.

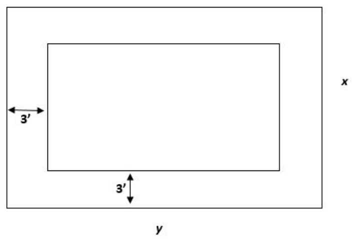
   Hence we get a minimum when $x = \sqrt{15}$. Now using $y = \frac{15}{x}$ we see $y = \sqrt{15}$.

2. Varying: The two numbers and the product
   Constant: The sum of one number and twice the other (20)
   Variable:
   Let x be one number
   Let y be the other
   Let P be the product
   Formula for the quantity we are trying to maximize: $P = xy$
   Formula for the constraint: $x + 2y = 20$
   Solve the constraint for one of the variables: $x = 20 - 2y$.
   Write the equation for P using one variable using substitution: $P = (20 - 2y)y = 20y - 2y^2$
   We need to create the number line for P'. $P' = 20 - 4y$. Hence $P' = 0$ when $y = 5$

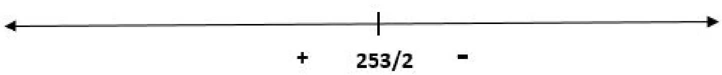
   

   Hence we get a minimum when $y = 5$. Now using $x = 20 - 2y$ we see $x = 10$.

3. Varying: The length and width of the outside rectangle (the inside rectangle is linked) and the area
Constant: The amount of fence (988)
Variable:
Let x be the length (see image below)
Let y be the width
Let A be the area of the outside rectangle

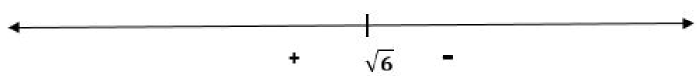


Formula for the quantity we are trying to maximize: $A = xy$
Formula for the constraint: $2x + 2y + 2(x - 6) + 2(y - 6) = 988$ which we can simplify to $4x + 4y - 24 = 988$
Solve the constraint for one of the variables: $y = \dfrac{1012 - 4x}{4} = 253 - x$.
Write the equation for A using one variable using substitution: $A = x(253 - x) = 253x - x^2$
We need to create the number line for A'. $A' = 253 - 2x$. Hence $A' = 0$ when $x = \frac{253}{2}$



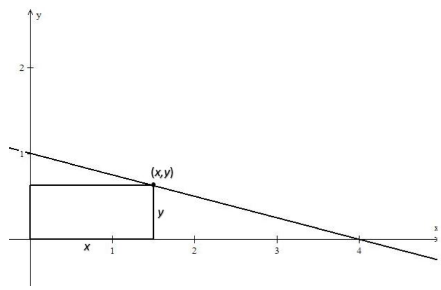
Hence we get the maximum area when $x = \frac{253}{2}$. Now using $y = 253 - x$ we see $y = \frac{253}{2}$.

4. Varying: The length, width, height and volume of the box.
Constant: The surface area of the box (36)
Variable:
Let x be the length of the base
Let $2x$ be the width of the base
Let y be the height of the box
Let V be the volume of the box
Formula for the quantity we are trying to maximize: $V = x(2x)y = 2x^2y$
Formula for the constraint: $2x^2 + 2xy + 2xy + xy + xy = 36$ Which simplifies to $2x^2 + 6xy = 36$
Solve the constraint for one of the variables: $y = \frac{36 - 2x^2}{6x}$.
Write the equation for V using one variable using substitution: $V = 2x^2 \frac{36 - 2x^2}{6x} = x\frac{36 - 2x^2}{3} = 12x - \frac{2}{3}x^3$
We need to create the number line for V'. $V' = 12 - 2x^2$. Hence $V' = 0$ when $x = \pm\sqrt{6}$ ... but only $\sqrt{6}$ makes sense.

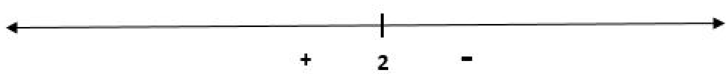


Hence we get the maximum volume when $x = \sqrt{6}$. Now using $y = \frac{36 - 2x^2}{6x}$ we see $y = \frac{4}{\sqrt{6}}$.

5. Varying: The length, width and area of the rectangle.
Constant: The rectangle's upper righthand corner must fall on the given line
Variable:
Let x be the length of the rectangle (and the x coordinate of the point on the line) (see labels in picture)
Let y be the width of the rectangle (and the y coordinate of the point on the line) (see labels in picture)
Let A be the area of the rectangle



Formula for the quantity we are trying to maximize: $A = xy$
Formula for the constraint: The equation of the given line is $y = -\frac{1}{4}x + 1$.
Solve the constraint for one of the variables: It already is
Write the equation for A using one variable using substitution: $A = x(-\frac{1}{4}x + 1) = -\frac{1}{4}x^2 + x$
We need to create the number line for A'. $A' = -\frac{1}{2}x + 1$. Hence $A' = 0$ when $x = 2$.



Hence we get the maximum area when $x = 2$. Now using $y = -\frac{1}{4}x + 1$ we see $y = \dfrac{1}{2}$.