IOP PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **28** (2007) 897–902 doi:10.1088/0143-0807/28/5/013

Golden ratio in a coupled-oscillator problem

Crystal M Moorman and John Eric Goff

School of Sciences, Lynchburg College, Lynchburg, VA 24501, USA

E-mail: goff@lynchburg.edu

Received 31 May 2007, in final form 14 June 2007
Published 19 July 2007
Online at stacks.iop.org/EJP/28/897

Abstract
The golden ratio appears in a classical mechanics coupled-oscillator problem that many undergraduates may not solve. Once the symmetry is broken in a more standard problem, the golden ratio appears. Several student exercises arise from the problem considered in this paper.

1. Introduction

The golden ratio emerges in a variety of contexts, from art and architecture to nature and mathematics. Examples where the golden ratio appears in physics include the study of the onset of chaos [1] and certain resistor networks [2]. Of the scores of books devoted to the golden ratio, we particularly enjoy Livio's book [3].

The irrational number $\varphi = (\sqrt{5}+1)/2$ ($\varphi = 1.618\,03\ldots$) is the golden ratio. Among its many interesting properties is that its inverse is related to itself by $\varphi^{-1} = \varphi - 1$ (i.e. $\varphi^{-1} = 0.618\,03\ldots$). We will show in the next section that the golden ratio and its inverse appear in the solution to the coupled oscillator problem illustrated in figure 1. Two identical masses are allowed to oscillate while connected to two identical massless springs. The oscillation takes place along a horizontal frictionless table. Finding normal modes and corresponding frequencies is our goal.

We have seen the aforementioned problem as a homework problem in Taylor's text [4], though he does not mention the golden ratio[1]. The problem also appears in other books [5, 6], except that it is turned 90° so that the masses and springs hang in a uniform gravitational field. While we obtain the same eigenfrequencies whether we solve the horizontal problem or the vertical problem, we find no mention of the golden ratio in the statements of the vertical problem.

While the problem we consider here is probably not solved that often by undergraduate and graduate classical mechanics students, they must surely solve one or both of the problems

[1] Taylor provides a back-of-the-book answer to his problem 11.5 on page 766. However, he writes the eigenfrequencies in the form of equation (7) instead of equation (9), and there is no mention of the golden ratio.

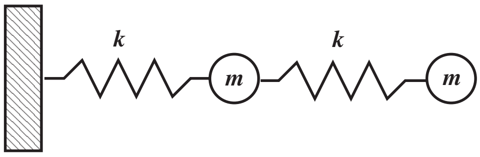

Figure 1. Our problem of interest. Both masses are identical as are both massless springs. The system oscillates on a horizontal frictionless table. The box on the left represents an immovable wall.

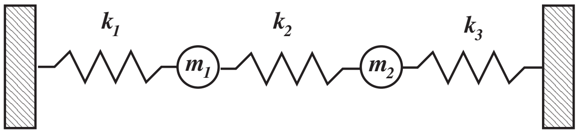

Figure 2. A paradigm of coupled oscillation seen in many textbooks. The boxes represent immovable walls. The problem shown here reduces to our problem of interest for $k_1 = k_2 = k$, $k_3 = 0$, and $m_1 = m_2 = m$.

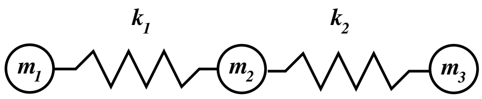

Figure 3. This textbook paradigm represents a model of a triatomic molecule (perhaps CO_2, carbon dioxide, if $m_1 = m_3$ and $k_1 = k_2$; perhaps HCN, hydrogen cyanide, if one is interested in modeling an asymmetric triatomic molecule).

shown in figures 2 and 3. Several classical mechanics textbooks [4–11] solve one or both of the aforementioned problems. If all masses in figures 2 and 3 are identical, as well as all the massless springs, both systems enjoy reflection symmetry. While the symmetry in the problem we solve is broken, we find it interesting that a number associated with aesthetic symmetry appears in the solution.

Section 2 will solve the problem given in figure 1 and section 3 will provide discussion for why the golden ratio appears in our solution. Section 4 will offer physics instructors some problem extensions that can be used in the classroom or for homework assignments.

2. Problem solution

The problem at hand is easily solved with either a Newtonian approach or a Lagrangian approach. We employ Newton's second law here. If the position of the left mass is x_1 and the position of the right mass is x_2, Newton's second law gives

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) \tag{1}$$

for the left mass and

$$m\ddot{x}_2 = -k(x_2 - x_1) \tag{2}$$

for the right mass. A dot represents a total time derivative. To solve the above coupled differential equations, we guess solutions of the form

$$x_1(t) = A_1\,\mathrm{e}^{\mathrm{i}\omega t} \tag{3}$$

and

$$x_2(t) = A_2\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{4}$$

where A_1 and A_2 are complex amplitudes and ω is a frequency to be determined.

Inserting equations (3) and (4) into equations (1) and (2), we find a matrix equation given by

$$\begin{pmatrix} -\omega^2 + 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5}$$

where $\omega_0^2 = k/m$. Note the asymmetry in the 2×2 matrix; namely, a factor of 2 appears in the top left entry while no such factor appears in the lower right entry. One can easily trace the origin of that factor of 2 back to the right-hand side of equation (1) where there are two $-kx_1$ terms. Physically, the factor of 2 arises because the left mass feels forces from two springs while the right mass only feels a force from one spring.

To avoid the trivial solution (i.e. $A_1 = A_2 = 0$), the determinant of the 2×2 matrix in equation (5) must vanish. The quadratic equation in ω^2 one gets is

$$(\omega^2)^2 - 3\omega_0^2(\omega^2) + \left(\omega_0^2\right)^2 = 0. \tag{6}$$

The two positive solutions are

$$\omega_\pm = \omega_0 \sqrt{\frac{3 \pm \sqrt{5}}{2}}, \tag{7}$$

where we label the two eigenfrequencies ω_+ and ω_-. The eigenfrequencies are simplified by noting that

$$3 \pm \sqrt{5} = \tfrac{1}{2}(\sqrt{5} \pm 1)^2, \tag{8}$$

meaning

$$\omega_\pm = \frac{\omega_0}{2}(\sqrt{5} \pm 1). \tag{9}$$

We now use the golden ratio ($\varphi = (\sqrt{5} + 1)/2$) and its inverse ($\varphi^{-1} = \varphi - 1$) to write the eigenfrequencies in the compact way

$$\omega_\pm = \varphi^{\pm 1}\omega_0. \tag{10}$$

Thus, the eigenfrequencies of the problem shown in figure 1 are related to the golden ratio in a very simple manner.

Going back to equation (5) and finding the relationship between A_1 and A_2 for each eigenfrequency allows us to find the normal modes. The (unnormalized) normal modes we find are

$$\eta_\pm = \varphi^{\pm 1}x_1 \mp x_2. \tag{11}$$

By the definition of normal modes, the decoupled differential equations satisfied by $\eta_\pm$ are

$$\ddot{\eta}_\pm + \omega_\pm^2 \eta_\pm = 0, \tag{12}$$

where $\omega_\pm$ are given by equation (10). Thus, the golden ratio appears in the normal modes, too. If the normal mode associated with $\omega_\pm$ is excited, the amplitude of the left mass is $\mp\varphi^{\pm 1}$ times the amplitude of the right mass.

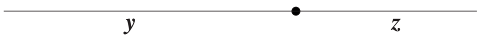

Figure 4. The golden section requires $(y+z)/y = y/z = \varphi$.

3. Discussion

To understand how the golden ratio enters into our problem, factor equation (6) as

$$\left(\omega^2 - \omega_0\omega - \omega_0^2\right)\left(\omega^2 + \omega_0\omega - \omega_0^2\right) = 0. \tag{13}$$

One of the two factors in the above equation must vanish. Setting the left factor to zero and solving for the positive root gives ω_+ from equation (9). If one instead looks for the positive root by equating the right factor to zero, one gets ω_- from equation (9).

Factoring equation (6) is where we make the connection with the golden ratio. Figure 4 shows the so-called golden section. What makes the section golden is that the ratio of $y+z$ to y is the same as the ratio of y to z. That ratio is defined as the golden ratio. Mathematically,

$$\frac{y+z}{y} = \frac{y}{z} = \varphi. \tag{14}$$

Setting up the quadratic equation for y gives

$$y^2 - zy - z^2 = 0. \tag{15}$$

The above equation's positive root is φz. Note that the form of the above equation is exactly the same as the left factor in equation (13). Had we wished to solve equation (14) as a quadratic equation for z, we would have obtained

$$z^2 + yz - y^2 = 0. \tag{16}$$

In other words, we obtain the exact same form as that of the right factor in equation (13). The above equation's positive root is $\varphi^{-1}y$.

Thus, the golden ratio enters our physical problem because the quadratic equation that one must solve to find the eigenfrequencies is exactly the same as the quadratic equation one must solve to obtain the golden ratio from the golden section. We thus see the golden ratio in our mechanical system of interest.

4. Classroom extensions

The first exercise an instructor might give a class is to derive equation (10). This is a standard undergraduate exercise in which a student makes use of ideas from linear algebra. A second exercise is to derive equation (11), or at least verify that after solving equation (11) for x_1 and x_2 in terms of η_+ and η_- and plugging back into equations (1) and (2) that equation (12) is the result. This last exercise is very straightforward, but it requires students to use an interesting property of the golden ratio, namely $\varphi^{-1} = \varphi - 1$.

If an instructor is looking for ways to incorporate computational techniques into a classical mechanics course, he or she could have students use symbolic software packages such as *Mathematica*, *MATLAB* and *JMathLib* and their linear algebra tools to derive the normalized normal modes, i.e. the normalized version of equation (11).[2] A simpler computational exercise is to impose initial conditions on the system and ask students to plot $x_1(t)$ and $x_2(t)$ versus

[2] Follow the procedure given in, for example, the text [8] by Thornton and Marion. See pp 475–85.

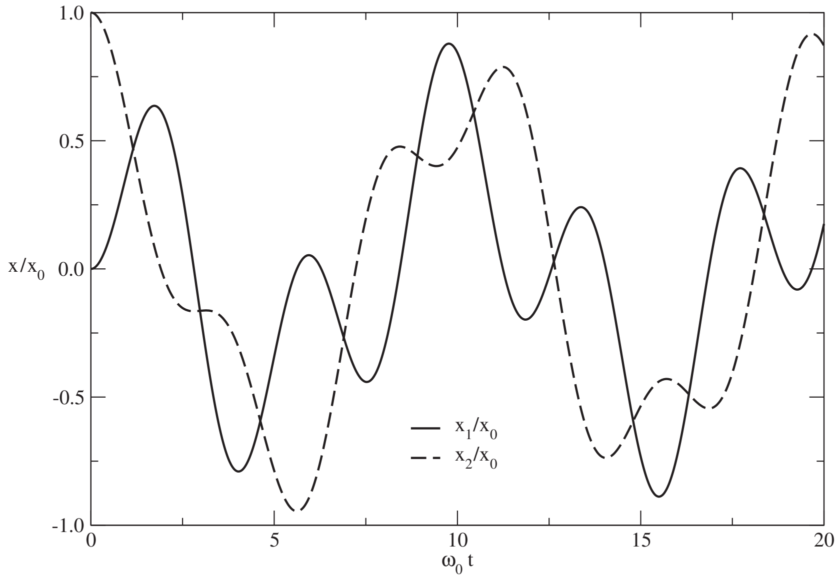

Figure 5. Equations (18) and (19) are plotted against the dimensionless time variable $\tilde{t} = \omega_0 t$.

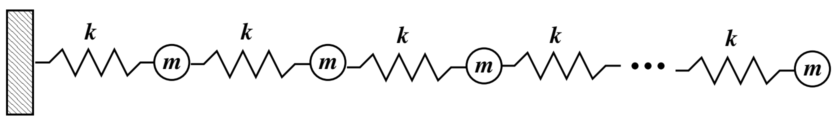

Figure 6. A challenge problem. Shown above are N identical masses on N identical massless springs.

time. For example, if the right mass in figure 1 is displaced to the right a distance x_0 from its equilibrium position while the left mass is held in its equilibrium position, the initial conditions that must be imposed are

$$x_1(0) = 0, \qquad x_2(0) = x_0, \qquad \dot{x}_1(0) = 0, \qquad \text{and} \qquad \dot{x}_2(0) = 0. \tag{17}$$

Students should then be able to derive the positions as functions of time to be

$$\tilde{x}_1(\tilde{t}) = \frac{1}{2\varphi - 1}[\cos(\varphi^{-1}\tilde{t}) - \cos(\varphi\tilde{t})] \tag{18}$$

and

$$\tilde{x}_2(\tilde{t}) = \frac{1}{2\varphi - 1}[\varphi\cos(\varphi^{-1}\tilde{t}) + \varphi^{-1}\cos(\varphi\tilde{t})]. \tag{19}$$

Here, we use dimensionless variables such that $\tilde{x}_1 = x_1/x_0$, $\tilde{x}_2 = x_2/x_0$ and $\tilde{t} = \omega_0 t$. Note also that $(2\varphi - 1)$ is just a fancy way of writing $\sqrt{5}$. Students can then plot the above equations and get what we show in figure 5. Note that there is never a complete transfer of energy from one mass to the other; i.e., one mass is never completely motionless at its equilibrium position.

Finally, a more challenging student exercise is to consider the problem illustrated in figure 6. That figure shows N identical masses and N identical massless springs. That system has a set of N eigenfrequencies $\{\omega_j\}_{j=1}^{N}$. Once students determine the N eigenfrequencies, they can show that

$$\prod_{j=1}^{N} \tilde{\omega}_j = 1, \tag{20}$$

where $\tilde{\omega}_j = \omega_j/\omega_0$. Students can check their results for the $N = 2$ case we examined in this paper[3]. Note from equation (10) that $\omega_+\omega_- = \omega_0^2$.

References

[1] Tabor M 1989 *Chaos and Integrability in Nonlinear Dynamics* (New York: Wiley) pp 162–7
[2] Srinivasan T P 1992 Fibonacci sequence, golden ratio, and a network of resistors *Am. J. Phys.* **60** 461–2
[3] Livio M 2002 *The Golden Ratio: The Story of Phi, the World's Most Astonishing Number* (New York: Broadway Books)
[4] Taylor J R 2005 *Classical Mechanics* (California: University Science Books) (see problem 11.5 on pp 448–9)
[5] Arya A P 1998 *Introduction to Classical Mechanics* (Englewood Cliffs, NJ: Prentice-Hall) (see problem 14.9 on pp 608–9)
[6] Chow T L 1995 *Classical Mechanics* (New Jersey: Wiley) (see problem 11.9 on p 431)
[7] Symon K R 1971 *Mechanics* (Longman, MA: Addison-Wesley) p 191
[8] Thornton S T and Marion J B 2004 *Classical Dynamics of Particles and Systems* (California: Thomson Brooks/Cole) pp 469 and 491
[9] Fowles G R and Cassiday G L 2005 *Analytical Mechanics* (California: Thomson Brooks/Cole) pp 472 and 496
[10] Goldstein H, Poole C and Safko J 2002 *Classical Mechanics* (New York: Addison-Wesley) p 253
[11] Fetter A L and Walecka J D 1980 *Theoretical Mechanics of Particles and Continua* (New York: McGraw-Hill) (see problem 4.2 on p 130)

[3] We offer students a couple of hints that may help them obtain equation (20). In the limit of large N, figure 6 looks like a loaded string with one end attached to a fixed wall. Follow the procedure in, for example, the text [8] by Thornton and Marion, beginning on p 498. (Thornton and Marion consider a loaded string fixed at both ends. While their problem is different than ours, the method of solution is not.) One can show that the eigenfrequencies can be written in terms of sine functions. To prove equation (20), try either an analytic proof or a geometric proof using a unit circle.