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Heuristic model of air drag on a sphere

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Abstract
A derivation of the drag coefficient for a sphere moving through air using a simple model employed by Newton is presented here. Assuming the air molecules are noninteracting and that they have elastic collisions with the sphere, the drag coefficient is $C_\mathrm{d} = 2$.

Introduction

My goal here is to evaluate the drag force on a sphere moving through air and determine the drag coefficient. While the heuristic model presented here is not new [1], the approach is rarely seen in introductory physics texts and books on classical mechanics, and I hope that instructors will find it useful in their classroom discussions of drag.

I start with a discussion of the basics of aerodynamic drag and then present the simple model of drag on a sphere. Finally, I conclude with discussions on how instructors could use the model in their classrooms and point out the model's deficiencies.

Background

Many introductory physics texts (e.g. [2]) and essentially all undergraduate mechanics texts (e.g. [3]) discuss the basics of drag forces on objects moving through fluids. Recent papers [4, 5] for teachers have examined air drag on spheres. The typical textbook tack is to investigate one-dimensional motion of an object moving through a fluid in which the drag force is proportional to the object's speed (Stokes force), v, or proportional to v^2, or a combination of the two. For objects like baseballs, the quadratic drag dominates linear drag.

The form of the drag force is often written as

$$F_\mathrm{D} = \tfrac{1}{2} C_\mathrm{d} \rho A v^2 \tag{1}$$

where ρ is the fluid density, A is the object's cross-sectional area and C_d is the 'drag coefficient'. The factor of $1/2$ is motivated by the fact that $\rho v^2/2$ is the 'dynamic pressure of the free stream' [6] while others, like White [7], use the factor of $1/2$ '…as a traditional tribute to Bernoulli and Euler…'. Students are introduced to dynamic pressure when they first meet fluid dynamics in an introductory course; so equation (1) should not be too mysterious. C_d obviously has no units and must be determined empirically. One could argue that C_d is really a 'fudge factor'; i.e., why not just determine F_D as a function of v experimentally and leave it at that? Why measure F_D for a particular v and then divide by $\rho A v^2/2$ to determine C_d? The idea is to use C_d to determine how the drag force *differs* from the free stream dynamic pressure multiplied by the object's cross-sectional area.

A heuristic model of the drag force on a sphere is presented in the next section. While actual drag is *much* more complicated than the model presented here, the model does give the correct functional dependences on ρ, A and v. It also provides a prediction for C_d that can be improved with better models.

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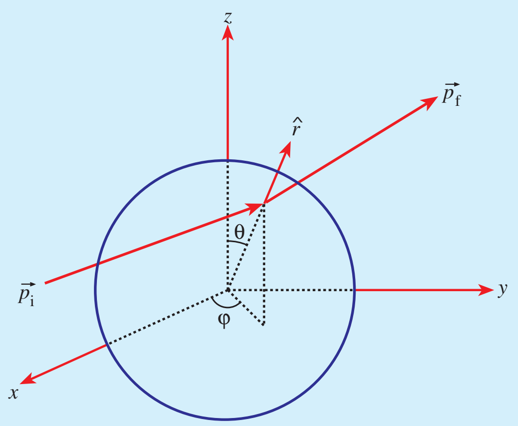

Figure 1. A stationary sphere with its centre at the origin of a Cartesian coordinate system. The air molecule is initially moving in the $-\hat{x}$ direction.

Heuristic model

Consider a sphere of radius R moving through initially stationary air with a speed v. Examine the sphere in a frame in which the sphere is at rest and the air is moving toward the sphere with a speed v. Consider a single air molecule (the compositional details are not important) of mass m_0 colliding with the sphere. To set the geometry of the problem, take the initial momentum of the air molecule to be $\vec{p}_{\rm i} = -m_0\, v\hat{x}$. The molecule collides with a stationary sphere whose centre coincides with the origin of a three-dimensional Cartesian coordinate system. See figure 1 and note that the azimuthal angle is φ and the polar angle is θ. The simple model assumes the air molecule has an *elastic* collision with the ball. Thus, the magnitude of the molecule's final linear momentum is equal to the magnitude of its initial linear momentum, i.e. $p_{\rm i} = p_{\rm f}$.

The force exerted on the air molecule by the ball is proportional to $\Delta\vec{p} = \vec{p}_{\rm f} - \vec{p}_{\rm i}$. The direction of $\Delta\vec{p}$ is along a radial line, or simply $\hat{r}$ (see figure 1). The magnitude of $\Delta\vec{p}$, $\mathcal{P}$, is found quite simply by examining the following:

$$\vec{p}_{\rm f} - \vec{p}_{\rm i} = \mathcal{P}\hat{r} \tag{2}$$

or

$$p_{{\rm f}x}\hat{x} + p_{{\rm f}y}\hat{y} + p_{{\rm f}z}\hat{z} + m_0 v_0\hat{x} = \mathcal{P}\left(\sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}\right). \tag{3}$$

Squaring the above equation gives

$$(p_{{\rm f}x} + m_0 v)^2 + p_{{\rm f}y}^2 + p_{{\rm f}z}^2 = \mathcal{P}^2. \tag{4}$$

Expanding equation (4) gives

$$p_{{\rm f}x}^2 + p_{{\rm f}y}^2 + p_{{\rm f}z}^2 + 2p_{{\rm f}x}m_0 v + (m_0 v)^2 = \mathcal{P}^2 \tag{5}$$

or

$$2p_{{\rm f}x}m_0 v + 2\,(m_0 v)^2 = \mathcal{P}^2. \tag{6}$$

Now, look at the $\hat{x}$ component of equation (3):

$$p_{{\rm f}x} + m_0 v = \mathcal{P}\sin\theta\cos\varphi. \tag{7}$$

Inserting $p_{{\rm f}x}$ from equation (7) into equation (6) gives

$$\mathcal{P} = 2m_0 v\sin\theta\cos\varphi. \tag{8}$$

Now,

$$\Delta\vec{p} = 2m_0 v\sin\theta\cos\varphi \times \left(\sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}\right). \tag{9}$$

The average value of $\Delta\vec{p}$ is found by integrating $\Delta\vec{p}$ over the sphere's area projected onto the y–z plane in figure 1 and dividing by the integral of that area projection. Thus,

$$\Delta\vec{p}_{\rm ave} = \frac{\int({\rm d}\vec{a}\cdot\hat{x})\Delta\vec{p}}{\int({\rm d}\vec{a}\cdot\hat{x})} \tag{10}$$

where ${\rm d}\vec{a} = R^2\sin\theta\,{\rm d}\theta\,{\rm d}\varphi\,\hat{r}$ is the surface element area vector and ${\rm d}\vec{a}\cdot\hat{x}$ is just the projection of ${\rm d}\vec{a}$ onto the y–z plane. The denominator of equation (10) is simply πR^2, the cross-sectional area of the sphere. With ${\rm d}\vec{a}\cdot\hat{x} = R^2\sin^2\theta\cos\varphi\,{\rm d}\theta\,{\rm d}\varphi$, equation (10) becomes

$$\Delta\vec{p}_{\rm ave} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}{\rm d}\varphi\cos\varphi\int_0^{\pi}{\rm d}\theta\,\sin^2\theta\,\Delta\vec{p}. \tag{11}$$

The integrals in equation (11) are trivial and the result is

$$\Delta\vec{p}_{\rm ave} = m_0 v\hat{x}. \tag{12}$$

The average force on a system of n air molecules that come into contact with the sphere in time Δt is just

$$\vec{F}_{\rm ave} = \frac{\Delta\vec{p}_{\rm ave}}{\Delta t}n. \tag{13}$$

n can be determined by considering a cylinder of air molecules colliding with the sphere in a time

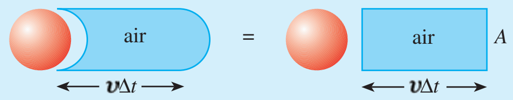

Figure 2. The figure on the left shows the 'real' air profile that hits the sphere. The volume of air is equivalent to the volume of air shown in the figure on the right, which is just a cylinder of length $\boldsymbol{v}\Delta t$ and end area A, the cross-sectional area of the sphere.

Δt. Figure 2 shows that the 'real' air volume is equivalent to the volume of a cylinder of length $v\Delta t$ and cross-sectional area A. The mass density of the air, ρ, is just the mass of the noninteracting air molecules divided by the volume of the cylinder of air shown in figure 2. Thus,

$$\rho = \frac{nm_0}{Av\Delta t} \tag{14}$$

or

$$n = \frac{\rho A v \Delta t}{m_0}. \tag{15}$$

Now, insert equations (12) and (15) into equation (13) to get

$$\vec{F}_{\rm ave} = \rho A v^2 \hat{x}. \tag{16}$$

Newton's third law would then simply say that the average force on the ball from the air has magnitude $\rho A v^2$ and points in the $-\hat{x}$ direction.

Discussion

Comparing equations (1) and (16), the heuristic model predicts that $C_{\rm d} = 2$. Note that, as discussed earlier, the real reason that equation (1) is used is to understand how the drag force *deviates* from a simple model's prediction. If $C_{\rm d}$ does not vary much with v, then a simple model adequately describes the physics. However, if $C_{\rm d}$ depends rather strongly on v, one would start to wonder if the simple model is even remotely touching the physics involved. Maybe $F_{\rm D}$ is indeed proportional to ρA, but the v^2 part could be seriously criticized.

On a first pass at trying to understand drag force, working through the simple model presented in the previous section is a good starting point, and equation (1) is a great way to begin discussions of more complex aspects of fluid mechanics. Investigations into vortices, boundary layers, boundary-layer separation etc lead to different values of $C_{\rm d}$.

One obvious shortcoming of the heuristic model presented here is that the air does not elastically bounce off the ball. Air actually flows around the ball and, at large enough speeds, forms vortices near the back of the sphere. For a spherical object like a baseball, the stitches affect the point at which the boundary layer becomes separated from the ball [8]. The presence of a boundary layer of fluid around the sphere and the delay in separation of that boundary layer due to surface roughness (stitching, for example) mean less drag and, consequently, a smaller drag coefficient. It should thus be of no surprise that the model presented here predicts too large a value for $C_{\rm d}$ for an object like a baseball [8]. (For small v, $C_{\rm d}$ is too large by about a factor of 4.)

Highly motivated students may wish to examine some of the theoretical extensions that could be made to the model here. A more sophisticated model might use the Navier–Stokes equation as a starting point.

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See Problem 9-13 on pages 360–1, particularly part (c). This problem partially motivates a solution using a model similar to the one in this work, though solution details are not given and there is no discussion of a drag coefficient.
[2] Halliday D, Resnick R and Walker J 2005 *Fundamentals of Physics* (New York: Wiley) pp 122–3
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Dodge quotes without proof a form of the drag analogous to equation (16).
[5] Eng J and Lietman T 1994 Measuring the velocity of a tennis serve *Phys. Teacher* **32** 168–72
[6] Olson R M 1965 *Essentials of Engineering Fluid Mechanics* (Scranton, PA: International Textbook) p 258

[7] White F M 1979 *Fluid Mechanics* (New York: McGraw-Hill) p 284

[8] Adair R K 2002 *The Physics of Baseball* (New York: Perennial/HarperCollins)
See page 8 and figure 2.1 for a plot of the drag coefficient versus ball velocity. See page 23 for the drag force Adair uses, which is $F_{\mathrm{D}} = C_{\mathrm{d}} A \rho \frac{v^2}{2}$. He claims that for '$\ldots C_{\mathrm{d}} = 2$, this is just the force required to move a column of air the size of the ball to match the velocity of the ball'. Thus, Adair's quoted value of C_{d} agrees with the one derived in this work.

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